

Compact 4-manifolds with harmonic Weyl tensor and nonnegative biorthogonal curvature

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Abstract

In this note we classify compact 4-manifolds with harmonic Weyl tensor and nonnegative biorthogonal curvature

M. S. C. (2000): 53C25, 53C24. **Key words:** Four-manifold, sectional curvature, biorthogonal curvature, harmonic curvature, harmonic Weyl tensor, Einstein manifold.

Let (M^4, g) be a Riemannian 4-manifold, s the scalar curvature of this metric g and K its sectional curvature. For each $p \in M^4$, let P a 2-plane in the tangent space $T_p M^4$ and $K^\perp(P)$ its orthogonal complement in $T_p M^4$. The *biorthogonal (sectional) curvature* relative to P (in $p \in M^4$) is the average

$$K^\perp(P) = \frac{K(P) + K(P^\perp)}{2} \quad [1.1]$$

Then we have the following function in M^4 :

$$K_1^\perp(p) = \inf\{K^\perp(P); P \subset T_p M^4\}, \quad [1.2]$$

If (M^4, g) be an oriented Riemannian 4-manifold, the Weyl tensor has the decomposition $W = W^+ \oplus W^-$, where $W^\pm : \Lambda^\pm \rightarrow \Lambda^\pm$ are self-adjoint with free traces and are called of the self-dual and anti-self-dual parts of W , respectively. Let $w_1^\pm \leq w_2^\pm \leq w_3^\pm$ be the eigenvalues of W^\pm , respectively. As was proved in [3] we have the following relation:

$$K_1^\perp - \frac{s}{12} = \frac{w_1^+ + w_1^-}{2}, \quad [1.3]$$

A Riemannian manifold (M, g) is said to have *harmonic curvature* if its Levi-Civita connection ∇ in the tangent bundle TM satisfies $d^*R = 0$, where R is the curvature tensor of (M, g) and d is the operator of exterior differentiation. If M is compact, this just means that ∇ is a critical point for the Yang-Mills functional $YM(\nabla) = \frac{1}{2} \int_M |R|^2$, where R is the curvature of connection ∇ .

Einstein metrics has harmonic curvature and metrics with harmonic curvature has constant scalar curvature .

Let (M^4, g) be an oriented Riemannian 4-manifold with self-dual Weyl tensor W^+ and anti-self-dual Weyl tensor W^- . Viewing W^\pm as a tensor of type $(0,4)$, we say that W^+ (W^-) is harmonic if $\delta W^+ = 0$ ($\delta W^- = 0$, respec.), where δ is the formal divergence defined for any tensor T of type $(0,4)$ by

$$\delta T(X_1, X_2, X_3) = -\text{trace}_g\{(Y, Z) \mapsto \nabla_Y T(Z, X_1, X_2, X_3)\},$$

where g is the metric of M . (M, g) has harmonic Weyl tensor W if W^\pm are harmonics. Einstein metrics and metrics with harmonic curvature are analytical metrics and has harmonic tensor Weyl.

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$$\delta T(X_1, X_2, X_3) = -\text{trace}_g\{(Y, Z) \mapsto \nabla_Y T(Z, X_1, X_2, X_3)\},$$

where g is the metric of M . (M^4, g) has harmonic Weyl tensor W if W^\pm are harmonics. Einstein metrics and metrics with harmonic curvature are analytical metrics and has harmonic tensor Weyl. In [1], R. Bettiol obtained the topological classification of compact simply connected 4-manifolds with positive biorthogonal and compacts 4-manifolds with harmonic Weyl tensor and positive biorthogonal was studied in [3] end [5]. In this note we prove the following

Theorem 1.1-*Let (M^4, g) be a compact oriented Riemannian 4-manifold with harmonic Weyl tensor, analytical metric and nonnegative biorthogonal curvature. Then (M^4, g) is an Einstein manifold with nonnegative sectional curvature, (M^4, g) is conformally flat or the universal covering of (M^4, g) is diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^2$ and g is a product of metrics with nonnegative sectional curvature.*

Proof

By the proof of Theorem 6 in [3], we have

$$|W^+| + |W^-| \leq \sqrt{6}[s/6 - 2K_1^\perp] \leq s/\sqrt{6},$$

where s is the scalar curvature of (M^4, g) .

Let $A = \{p \in M^4; Ric(p) \neq s(p)/4\}$ and $B = \{p \in M^4; |W^+| = |W^-|\}$.

Note that $A \subset B$. If A is empty then (M^4, g) is an Einstein manifold. Otherwise we deduce that $B = M^4$ and using the Weitzenböck formula for W^\pm , is easy see that $W^\pm = 0$ or $|W^+|^2 = |W^-|^2 = s^2/24$, $w_1 \pm = w_2^\pm$ and so (M^4, g) has nonnegative isotropic curvature. In this case we can use the theorem B in [2] (see also Prop. 3 in [4])

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